

TRANSMUTATION METHOD FOR SOLVING HYPER-BESSEL DIFFERENTIAL EQUATIONS BASED ON THE POISSON-DIMOVSKI TRANSFORMATION

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*Dedicated to the 75th anniversary
of Professor Gary Roach*

Abstract

A generalization of the Poisson-type integral transformation proposed by Dimovski is applied to hyper-Bessel differential equations of arbitrary order, as a transmutation operator. By this method, their solutions are written in an explicit form, by evaluating operators of the generalized fractional calculus of generalized trigonometric functions. A differential analogue of the Poisson-Dimovski transformation is also proposed and its applications to "spherical" hyper-Bessel equations from Kamke's book are illustrated.

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1. Hyper-Bessel differential operators and equations

The second order *differential operator of Bessel*

$$B_\nu = \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \frac{\nu^2}{x^2} = x^{-2} \left(x \frac{d}{dx} + \nu \right) \left(x \frac{d}{dx} - \nu \right) = x^{-\nu-1} \frac{d}{dx} x^{2\nu+1} \frac{d}{dx} x^{-\nu}, \quad (1)$$

and its modification, the so-called *Bessel-Clifford (or Weinstein) operator*

$$\widehat{B}_\nu = \frac{d^2}{dx^2} + \frac{2\nu+1}{x} \frac{d}{dx} = x^{1-\nu} B_\nu x^\nu, \quad (2)$$

are related to the *Bessel function* $y(x) = J_\nu(x)$ as a solution of the Bessel equation $B_\nu y(x) = -y(x)$. Operators (1),(2) and all their variants related

to the modified Bessel and the so-called cylindrical functions, play an important role in modeling various problems of mathematical physics and engineering practice, and might be considered as giving rise to the topic called “*Special Functions of Mathematical Physics*”.

In a series of papers, starting from 1966 (see for example [5], [6], etc.) Dimovski introduced a very general class of linear singular differential operators B of arbitrary order $m \geq 2$, with variable coefficients, coming as generalizations of (1) and (2) and called them *Bessel-type operators*. Nowadays, their name “*hyper-Bessel operators*” became widely popular. For this class of operators, he developed Mikusinski’s type operational calculi (OC) based on the notion (introduced by him, see [8]) “*convolution of linear operator*”. More precisely, in this case, the convolution and the corresponding OC are for the linear integral operator L - right inverse to B ($BL = Id$) under zero initial conditions of Bessel type. Dimovski suggested also an alternative approach to OC for the hyper-Bessel operators (integral and differential) based on a generalization of the Laplace transform, called *Obrechkoﬀ transform* (see [6], [11]; whole details in Ch.3, §3.9, §3.10 in [15]).

DEFINITION 1. Let $m \geq 1$ be integer, $\beta > 0$; $\gamma_1, \gamma_2, \dots, \gamma_m$ be given real parameters. Under *hyper-Bessel differential operator* we mean each linear singular differential operator of m -th order with variable coefficients, of one of the equivalent forms

$$B = x^{-\beta} \prod_{k=1}^m \left(x \frac{d}{dx} + \beta \gamma_k \right) = x^{-\beta} Q_m \left(\frac{d}{dx} \right), \text{ with } m\text{-degree polynomial } Q_m, \quad (3)$$

or

$$B = x^{-\beta} \left[x^m \frac{d^m}{dx^m} + a_1 x^{m-1} \frac{d^{m-1}}{dx^{m-1}} + \dots + a_{m-1} x \frac{d}{dx} + a_m \right], \quad (4)$$

or

$$B = x^{\alpha_0} \frac{d}{dx} x^{\alpha_1} \frac{d}{dx} \dots x^{\alpha_{m-1}} \frac{d}{dx} x^{\alpha_m}. \quad (5)$$

For the sake of technical simplicity, the zeros of the polynomial Q_m in (3), $\mu_k = -\beta \gamma_k, k = 1, \dots, m$ are supposed real, and the parameters γ_k arranged, say as: $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_m$, hence $\alpha := \max_k [-\beta(\gamma_k + 1)] = -\beta(\gamma_m + 1)$. When working on the real half-line (variable $x \in [0, +\infty)$), the hyper-Bessel differential operators are usually considered on the function spaces of continuous, resp. continuously-differentiable functions with respect to power weights, as:

$$C_\alpha^{(k)} := \left\{ f(x) = x^p \tilde{f}(x); p > \alpha, \tilde{f} \in C^{(k)}[0, \infty) \right\}, \text{ where } C_\alpha^{(0)} := C_\alpha.$$

In $C_{\alpha+\beta}^{(m)}$, the differential operator B is right-invertible and its linear right inverse operator $y(x) := Lf(x)$, defined as the solution of the IVP

$$By(x) = f(x), \quad \lim_{x \rightarrow +0} B_k y(x) = 0, \quad k = 1, \dots, m, \quad (6)$$

with $B_k := x^{-\beta\gamma_k} \prod_{j=k+1}^m \left(x \frac{d}{dx} + \beta\gamma_j \right) = x^{\alpha_k} \frac{d}{dx} x^{\alpha_{k+1}} \frac{d}{dx} \dots x^{\alpha_m},$

has the following representation in the works of Dimovski (as [5], [6]):

$$Lf(x) = \beta^{-m} x^\beta \int_0^1 \dots \int_0^1 \left(\prod_{k=1}^m x_k^{\gamma_k} \right) f \left[x(x_1 \dots x_m)^{1/\beta} \right] dx_1 \dots dx_m. \quad (7)$$

We call (7) a *hyper-Bessel integral operator*. Obviously, $B : C_{\alpha+\beta}^{(m)} \longrightarrow C_\alpha$, but $L : C_\alpha \longrightarrow C_{\alpha+\beta}^{(m)} \subset C_\alpha$.

DEFINITION 2. The ordinary differential equations of the form

$$By(x) - \lambda y(x) = f(x), \quad \lambda = \text{const}, \quad f(x) \in C_\alpha, \quad (8)$$

are called *hyper-Bessel differential equations*.

Typical examples of the operators (3), (4), (5) are the 2nd order Bessel differential operators (1),(2), thus suggesting the new notion in Def. 1. Let us mention also the most simple m -th order hyper-Bessel operator: the m -th order differentiation $D^m = (d/dx)^m$. Thus, the Bessel differential equation $B_\nu y(x) = -y(x)$, and the differential equations $D^m y(x) = y^{(m)} = \lambda y(x)$, $m \geq 2$, are examples of hyper-Bessel ODEs (8). A solution to the first equation is given by the Bessel function

$$J_\nu(x) = (x/2)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j}}{j! \Gamma(j + \nu + 1)} \quad (9)$$

$$= \frac{(x/2)^\nu}{\Gamma(\nu + 1)} {}_0F_1(\nu + 1; -(x/2)^2) := \frac{(x/2)^\nu}{\Gamma(\nu + 1)} j_\nu(x),$$

where $j_\nu := {}_0F_1$ is called also *normalized Bessel function*. The sets of the m independent solutions of the latter m -th order equations are known as *generalized m -th order trigonometric (resp. hyperbolic) functions*: $\cos_m(x)$ and $\sin_{k,m}$, $k = 1, \dots, m-1$ for $\lambda = -1$; and resp. $h_{1,m}(x)$, $k = 1, \dots, m$ for $\lambda = +1$ (for definitions and details, see [12], Vol. 3, Ch. 18; [15], Appendix; etc). For example, the *generalized cosine function of order m*

$$\cos_m(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{mj}}{(mj)!}, \quad m \geq 2, \quad (10)$$

reduces for $m = 2$ to the classical cosine function $\cos(x) = \cos_2(x)$ as a solution of $y''(x) = -y(x)$.

For the Bessel function, the following integral representation

$$J_\nu(x) = \frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^{\pi/2} \cos(x \sin \varphi) (\cos \varphi)^{2\nu} d\varphi, \quad \Re \nu > -1/2, \quad (11)$$

is known as the *Poisson integral (formula)*, see e.g. [12], Vol. 2, §7.12. After a substitution $\sin \varphi := t$, it gets the form of a fractional order $(\nu+1/2)$ operator of integration (*Erdélyi-Kober operator of Riemann-Liouville type*):

$$J_\nu(x) = \frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(xt) dt. \quad (12)$$

In our studies (as [10], [15, Chs. 3, 4], [16]) on hyper-Bessel differential equations (8) and their solutions, expressed in terms of the hyper-Bessel functions $j_{\nu_1, \dots, \nu_{m-1}}^{(m-1)}(x)$ of Delerue [2], as analogues of the normalized Bessel function $j_\nu(x)$ in (9), we have introduced generalizations of the Poisson integral formula (12) of $j_{\nu_1, \dots, \nu_{m-1}}^{(m-1)}(x)$ involving the generalized cosine function (10) of order m via operators of generalized fractional calculus, [15].

2. Transmutation method and generalized Poisson transformation due to Dimovski

The essence of this method lies in the natural striving to find solutions of new complicated problems by their reduction to well-known or simpler ones, by means of a specific “*translator*”. In a narrow sense, the notion of *transmutation operator* originates from the works of Delsarte and Lions (1956 - 1959), see for example [4] and the posthumously published works [3], Vol. 1, p. 427. If $\tilde{B} : \mathcal{X} \longrightarrow \mathcal{X}$ and $B : \mathcal{X} \longrightarrow \mathcal{X}$ are two operators acting in a space \mathcal{X} , then the isomorphism $T : \mathcal{X} \longrightarrow \mathcal{X}$ is said to *transmute \tilde{B} into B* , if the similarity relation holds: $T\tilde{B} = BT$ in \mathcal{X} . In this sense, the transmutation method has been widely used in mathematical analysis, and mainly in solving differential equations and problems of mathematical physics (see Delsarte [3], Delsarte and Lions [4], Hearsh [13]. For applications in operational calculus, extended to so-called convolutional calculi, see for example Dimovski [6], [7], [8]. Some of the authors use “similarities” (or “transmutations”) in a wider sense, as below.

DEFINITION 3. (Dimovski [8]) An isomorphism $T : \tilde{\mathcal{X}} \longrightarrow \mathcal{X}$ of the linear space \mathcal{X} into another linear space \mathcal{X} , is said to be a *similarity (similarity operator, transmutation operator)* from a linear operator $\tilde{L} : \tilde{\mathcal{X}} \longrightarrow \tilde{\mathcal{X}}$ to the linear operator $L : \mathcal{X} \longrightarrow \mathcal{X}$, if $T\tilde{L} = LT$ holds in $\tilde{\mathcal{X}}$. We say also that *the operator \tilde{L} is similar to L under similarity T .*

The *similarity relation*

$$T\tilde{L} = LT \quad \text{in } \tilde{\mathcal{X}} \quad (13)$$

can be written also in the forms

$$\tilde{L} = T^{-1}LT \quad \text{in } \tilde{\mathcal{X}}, \quad \text{or } L = T\tilde{L}T^{-1} \quad \text{in } \mathcal{X}. \quad (14)$$

DEFINITION 4. ([8]) If the operators \tilde{L} and L are the corresponding linear right inverses of the right-invertible operators $\tilde{B} : [\tilde{\mathcal{X}}_B \subset \tilde{\mathcal{X}}] \longrightarrow \tilde{\mathcal{X}}$ and $B : [\mathcal{X}_B \subset \mathcal{X}] \longrightarrow \mathcal{X}$, we say also that *the operator T transmutes (transforms) \tilde{B} into B* (in a wider sense), even if the similarity relation $T\tilde{B} = BT$ may not be fulfilled exactly (up to additional terms depending on the initial conditions).

The Poisson integral representation (12) of the Bessel function was a starting point for Delsarte [3], Vol. 1, p. 439, to introduce the *Poisson transformation (transmutation operator)*

$$P_\nu f(x) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} f(xt) dt. \quad (15)$$

This operator transmutes the simplest 2nd order differential operator $D^2 = (d/dx)^2$ into the Bessel (more exactly Bessel-Clifford) operator (2), that is, the similarity $P_\nu (d/dx)^2 = \hat{B}_\nu P_\nu$ holds in a corresponding functional space.

A more general task can be considered: to find transmutation operators in the sense of Def. 4, when the linear right-invertible operators \tilde{B} and B are *two arbitrary hyper-Bessel differential operators of the same order $m > 1$ with different parameters*. While developing operational calculi, and to find corresponding convolution operations for the hyper-Bessel operators, in a series of papers (for example [6],[7]) Dimovski solved this difficult problem. He proposed a family of very general integral transformations T of Poisson-Sonine (P-S) type. Nowadays, these are called *Poisson-Sonine-Dimovski (P-S-D) transformations (transmutations)*. More details can be seen also in Dimovski [7], Dimovski and Kiryakova [9],[11] as well as in Kiryakova [15], Ch. 3, §3.5. Further, he specified one of these two operators to be

the simplest hyper-Bessel differential operator of order m , say $\tilde{B} := D^m = (d/dx)^m$. When T has to transmute the arbitrary m -th order Bessel-type differential operator B into $\tilde{B} = D^m$, resp. to be a similarity between the right-inverse integral operators L (of B) and I^m (the m -fold integration, of D^m): $TL = I^m T$, these are the Sonine-type transformations, called *Sonine-Dimovski transformations*. They are coming as natural generalizations of the *Sonine transformation*, considered by Delsarte [3], Vol. 1, and inspired by the well-known Sonine integral (formula) for the Bessel functions:

$$S_\nu f(x) = \frac{2(\nu/2)^{\nu+1}}{\Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} t^{1-\nu} f(xt) dt.$$

Both integral transformations P_ν and S_ν of Poisson and Sonine, are seen to represent special cases of the *classical operators of fractional integration* of order $\nu + 1/2$ in the sense of Erdélyi-Kober, see [21], or [15], Ch.2.

Conversely, when T transmutes D^m into arbitrary m -th order hyper-Bessel differential operator B , resp. T is a similarity between the integral operators I^m and L : $TI^m = LT$, these are the Poisson-type transformations, called Poisson-Dimovski transformations. For the "hyper-Bessel" differential operator $\tilde{B} = D^m = (d/dx)^m$, of the form (5) with $\tilde{\alpha}_0 = \tilde{\alpha}_1 = \dots = \tilde{\alpha}_m = 0$, the alternative representation (3) is with parameters: $\tilde{\beta} = m$; $\tilde{\gamma}_k = \frac{k}{m} - 1$, $k = 1, \dots, m$, therefore $\tilde{\gamma}_1 < \tilde{\gamma}_2 < \dots < \tilde{\gamma}_m = 0$, $\tilde{\alpha} = -1$. Its linear right inverse operator, the operator of m -fold integration $\tilde{L} = I^m$, has therefore also the alternative representations

$$\begin{aligned} \tilde{L}f(x) &:= I^m f(x) = \int_0^x dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} f(t_m) dt_m \\ &= \frac{x^m}{m^m} \int_0^1 \dots \int_0^1 x_1^{-\frac{m-1}{m}} \dots x_m^{-\frac{1}{m}} f \left[x(x_1 \dots x_m)^{1/m} \right] dx_1 \dots dx_m. \end{aligned} \quad (16)$$

DEFINITION 5. Let

$$\Delta_k := \tilde{\gamma}_k - \gamma_k = \frac{k}{m} - 1 - \gamma_k, \quad k = 1, \dots, m; \quad \Delta := \max_{1 \leq k \leq m} \Delta_k > \frac{\alpha}{\beta} + \frac{1}{m}. \quad (17)$$

The integral transformation $\mathcal{P} : C_{-1} \longrightarrow C_\alpha$, defined as

$$\begin{aligned} \mathcal{P}f(x) &= c \left(\frac{x^\beta}{\beta^m} \right)^\Delta \int_0^1 \dots \int_0^1 \prod_{k=1}^m \left[\frac{(1-t_k)^{\Delta-\Delta_k-1}}{\Gamma(\Delta-\Delta_k)} t_k^{\tilde{\gamma}_k} \right] \\ &\quad \times f \left[\frac{m}{\beta} x^{\beta/m} (x_1 \dots x_m)^{1/m} \right] dx_1 \dots dx_m \end{aligned}$$

$$\begin{aligned}
 &= c \left(\frac{x^\beta}{\beta^m} \right)^\Delta \int_0^1 \cdots \int_0^1 \prod_{k=1}^m \left[\frac{(1-t_k)^{\Delta+\gamma_k-\frac{k}{m}}}{\Gamma(\Delta+\gamma_k-\frac{k}{m}+1)} t_k^{\frac{k}{m}-1} \right] \\
 &\quad \times f \left[\frac{m}{\beta} x^{\beta/m} (x_1 \dots x_m)^{1/m} \right] dx_1 \dots dx_m \quad (18)
 \end{aligned}$$

is called a *Poisson-Dimovski transformation (transmutation)*, corresponding to the hyper-Bessel operator (3)-(5). Here, and in what follows, we have used for convenience the scalar multiplier

$$c := \sqrt{m/(2\pi)^{m-1}}.$$

Dimovski [6],[7] proved that

THEOREM 1. *Transformation (18) is a similarity from $\tilde{L} = I^m$ to L in the space C_{-1} , that is, $\mathcal{P}I^m f(x) = L\mathcal{P}f(x)$, $f \in C_{-1}$.*

In a wider sense, this means that \mathcal{P} transmutates the m -th order differentiation D^m into the m -th order hyper-Bessel differential operator B . Indeed, the following corollary holds (given as Theorem 2 in Dimovski and Kiryakova [10]; see also Lemma 3.5.8 and Corollary 3.5.9 in Kiryakova [15], Ch. 3, §3.5, p. 139):

COROLLARY 2. *In the functional space $C_{\tilde{F}} := \text{span}\{x^k\}_0^{m-1} \oplus C_{m-1}^{(m)}$, we have*

$$\mathcal{P} \left(\frac{d}{dx} \right)^m f(x) = B \mathcal{P}f(x) - B \mathcal{P} \tilde{F}f(x), \quad (19)$$

where \tilde{F} stands for the initial values operator (so-called, projector operator, see def. in Dimovski [8]) of the operator $\tilde{L} = I^m$, which has the form of the Taylor polynomial of f in $x = 0$:

$$\tilde{F}f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} x^k.$$

COROLLARY 3. ([15], Cor. 3.5.9, p. 139) *On the subspace $[\mathcal{X}_{\tilde{B}} = C_{m-1}^{(m)}] \subset \mathcal{X}_{\tilde{F}}$, there is a similarity relation also between the differential operators:*

$$\mathcal{P} \left(\frac{d}{dx} \right)^m = B \mathcal{P}. \quad (20)$$

In particular, the above equality holds true for the so-called *m-even functions* (see Ključantčev [19]): $f \in C_{m-1}^{(m)} \subset C^{(m)}$, $f'(0) = \dots = f^{(m-1)}(0) = 0$. For $m = 2$, an 2-even function is the cosine: $\cos_2(x) = \cos x$.

In our subsequent works [9], [10], [15], etc, we have interpreted the Poisson- and Sonine-type transformations introduced by Dimovski, as operators of the generalized fractional calculus, giving alternative representations for them by means of single integrals involving special functions as kernels.

3. Notions of generalized fractional calculus (GFC), related generalized hypergeometric functions

Here we give only a brief account on some special functions used in this survey and on the *operators of Generalized Fractional Calculus (GFC)*, introduced and studied by Kiryakova [15] and in a series of papers like [9],[10], [16]. Short survey on these operators, their origin and operational properties can be seen in the recent survey Kiryakova [18], that is also available on-line at http://www.math.bas.bg/~fcaa/volume11/fcaa112/Kiryakova_fcaa112.pdf and http://www.diogenes.bg/fcaa/volume11/fcaa112/Kiryakova_fcaa112.pdf.

DEFINITION 6. Let $m \geq 1$ be integer, $\beta > 0$, $\gamma_1, \dots, \gamma_m$ and $\delta_1 \geq 0, \dots, \delta_m \geq 0$ be real parameters. By a *generalized (multiple, m -tuple) operator of integration of fractional multi-order $\delta = (\delta_1, \dots, \delta_m)$* we mean an integral operator of the form

$$I_{\beta, m}^{(\gamma_k), (\delta_k)} f(x) = \int_0^1 G_{m, m}^{m, 0} \left[\sigma \left| \begin{matrix} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix} \right. \right] f(x\sigma^{\frac{1}{\beta}}) d\sigma. \quad (21)$$

Also, each operator of the form

$$\mathcal{R}f(x) = x^{\beta\delta_0} I_{\beta, m}^{(\gamma_k), (\delta_k)} f(x) \quad \text{with arbitrary } \delta_0 \geq 0,$$

is said to be a *generalized (m -tuple) operator of fractional integration of Riemann-Liouville type*, or briefly: a *generalized (R.-L.) fractional integral*.

Operator (21) is a typical representative of the so-called “*generalized operators of fractional integration*” having the general form (introduced by Kalla, 1970-1979)

$$If(x) = \int_0^1 \Phi(\sigma) \sigma^\gamma f(x\sigma) d\sigma,$$

with a suitable special function $\Phi(\sigma)$ as a kernel-function. In our case, the kernel-function $G_{m, m}^{m, 0}$ is a specific case of the Meijer’s G -function, such to allow a general theory of GFC combined with useful applications.

DEFINITION 7. (see [12], Vol.1; [20]; [15], Appendix) By a *Meijer’s G -function*, we mean the generalized hypergeometric function defined by means of the contour integral in the complex plane

$$G_{p,q}^{m,n}(\sigma) = G_{p,q}^{m,n} \left[\sigma \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{G}_{p,q}^{m,n}(s) \sigma^s ds, \quad (22)$$

where the integrand in (22) has the structure

$$\mathcal{G}_{p,q}^{m,n}(s) = \frac{\prod_{k=1}^m \Gamma(b_k - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{k=m+1}^q \Gamma(1 - b_k + s) \prod_{j=n+1}^p \Gamma(a_j - s)},$$

\mathcal{L} is a suitable contour in \mathbb{C} ; the orders $(m, n; p, q)$ are nonnegative integers such that $0 \leq m \leq q$, $0 \leq n \leq p$; and the parameters $a_j, j = 1, \dots, p$, $b_k, k = 1, \dots, q$ are arbitrary complex numbers such that

$$(b_k + l) \neq (a_j - l' - 1); \quad l, l' = 0, 1, 2, \dots; \quad j = 1, \dots, p, \quad k = 1, \dots, q.$$

For further extensions of the generalized fractional integrals (21), involving the so-called *Fox's H-function*, see for example in Kiryakova [15], Ch.5; [18], etc.

Meijers' G -functions are useful tools because almost all the special functions of mathematical physics, as well as the basic elementary functions, can be represented as G -functions. Especially, the *generalized hypergeometric functions* (*ghf-s*) ${}_pF_q(s)$ which denotations we shall need in this paper, are the most often used G -functions (see more in [12], Vol.1; [20]; [15], App.):

$$\begin{aligned} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; s) &= \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{s^k}{k!} \\ &= \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G_{p,q+1}^{1,p} \left[-s \left| \begin{array}{c} 1 - a_1, \dots, 1 - a_p \\ 0, 1 - b_1, \dots, 1 - b_q \end{array} \right. \right], \quad (23) \end{aligned}$$

where $p \leq q$ and $s \in \mathbb{C}$, or $p = q + 1$ and $|s| < 1$, and the Pochhammer symbol denotes $(a)_0 = 1$, $(a)_k = \Gamma(a + k)/\Gamma(a) = a(a + 1) \dots (a + k - 1)$.

Examples of ghf-s (23), discussed here, are the (normalized) Bessel functions $j_\nu(x) = {}_0F_1(-; \nu + 1; -(x/2)^2)$, (9); the generalizes trigonometric and hyperbolic functions; the hyper-Bessel functions, the Gauss function, etc.

Our generalized operators of integration of fractional multi-order (21) are natural extension of the fractional integration operators in the classical

fractional calculus (FC). For $m = 1$ the kernel G -function is an elementary function and we obtain the *Erdélyi-Kober (E-K) fractional integration operators*:

$$I_{\beta}^{\gamma, \delta} f(x) = \int_0^1 \frac{(1-\sigma)^{\delta-1} \sigma^{\gamma}}{\Gamma(\delta)} f(x\sigma^{\frac{1}{\beta}}) d\sigma, \quad \gamma \in \mathbb{R}, \beta > 0, \quad (24)$$

and in particular, the *Riemann-Liouville (R-L) fractional integrals* (of order $\delta > 0$): $R^{\delta} f(x) = x^{\delta} I_1^{0, \delta} f(x)$. For $m = 2$, the $G_{2,2}^{2,0}$ -kernel reduces to the Gauss hypergeometric function ${}_2F_1(1-s)$, and thus we have the so-called *fractional hypergeometric integral operators* of Love, Saxena, Kalla, Saigo et al. Many other special cases of operators of FC are also following from the GFC operators, as illustrated in [15]. The key to the numerous applications of the operators (21) is hidden in the fact that besides the single integral representation in Def. 6, they can be presented also as *commutative compositions of classical E-K operators* (24), without any use of special functions in the kernel. Namely (see Th. 7 in [18]),

$$\begin{aligned} I_{\beta, m}^{(\gamma_k), (\delta_k)} f(x) &= I_{\beta_m}^{\gamma_m, \delta_m} \left\{ I_{\beta_{m-1}}^{\gamma_{m-1}, \delta_{m-1}} \dots \left(I_{\beta_1}^{\gamma_1, \delta_1} f(x) \right) \right\} = \left[\prod_{k=1}^m I_{\beta_k}^{\gamma_k, \delta_k} \right] f(x) \\ &= \int_0^1 \dots \int_0^1 \left[\prod_{k=1}^m \frac{(1-\sigma_k)^{\delta_k-1} \sigma_k^{\gamma_k}}{\Gamma(\delta_k)} \right] f \left(x \sigma_1^{\frac{1}{\beta_1}} \dots \sigma_m^{\frac{1}{\beta_m}} \right) d\sigma_1 \dots d\sigma_m. \end{aligned} \quad (25)$$

By analogy with the classical FC and the R-L fractional derivatives, we introduce also the generalized operators of differentiation of fractional multi-order $(\delta_1, \dots, \delta_m)$, as interpretations by means of integro-differential operators, of the symbols in (21) when not all components of the multi-order are nonnegative!

DEFINITION 8. With the same parameters as in Def. 6 and the integers

$$\eta_k = \begin{cases} \delta_k, & \text{if } \delta_k \text{ is integer,} \\ [\delta_k] + 1, & \text{if } \delta_k \text{ is noninteger,} \end{cases} \quad k = 1, \dots, m, \quad (26)$$

we introduce the auxiliary differential operator

$$D_{\eta} = \left[\prod_{r=1}^m \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta} x \frac{d}{dx} + \gamma_r + j \right) \right]. \quad (27)$$

Then, we define the *generalized (multiple, m -tuple) fractional derivative* of multi-order $\delta = (\delta_1 \geq 0, \dots, \delta_m \geq 0)$ by means of the differ-integral operator:

$$\begin{aligned}
 D_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) &= D_\eta I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(x) \\
 &= \left[\prod_{r=1}^m \prod_{j=1}^{\eta_r} \left(\frac{1}{\beta} x \frac{d}{dx} + \gamma_r + j \right) \right] I_{\beta,m}^{(\gamma_k+\delta_k),(\eta_k-\delta_k)} f(x).
 \end{aligned} \tag{28}$$

More generally, all differ-integral operators of the form

$$\mathcal{D}f(x) = D_{\beta,m}^{(\gamma_k),(\delta_k)} x^{-\delta_0} f(x) = x^{-\delta_0} D_{\beta,m}^{(\gamma_k-\frac{\delta_0}{\beta}),(\delta_k)} f(x) \quad \text{with } \delta_0 \geq 0, \tag{29}$$

are also called *generalized (multiple, multi-order) fractional derivatives*.

The hyper-Bessel differential and integral operators happen to be interesting examples of the operators of GFC, for arbitrary $m \geq 1$ and multi-order of differentiation/ resp. integration $\delta = (1, 1, \dots, 1)$, that is with $\forall \delta_k = 1, k = 1, \dots, m$. Indeed, as we have shown yet in [9], see details in [15], Ch.3; [11], for the operators B and L we have the alternative (GFC) representations in terms of the denotations $D_{\beta,m}^{(\gamma_k),(\delta_k)}$ and $I_{\beta,m}^{(\gamma_k),(\delta_k)}$:

$$Bf(x) = \beta^m x^{-\beta} D_{\beta,m}^{(\gamma_k-1), (1)} f(x) = \beta^m D_{\beta,m}^{(\gamma_k), (1)} x^{-\beta} f(x), \tag{30}$$

$$Lf(x) = \beta^{-m} x^\beta I_{\beta,m}^{(\gamma_k), (1)} f(x), \quad \left[\text{i.e. for } \beta = m : L = \left(\frac{x}{m} \right)^m I_{m,m}^{(\gamma_k), (1)} \right], \tag{31}$$

whence by the operational rules of the GFC ([15], [18]), it is easily seen that $BL = \left(\beta^m D_{\beta,m}^{(\gamma_k), (1)} x^{-\beta} \right) \left(\beta^{-m} x^\beta I_{\beta,m}^{(\gamma_k), (1)} \right) = Id$, the identity in C_α .

Among the hyper-Bessel operators, we shall further use *GFC representations* (that is, in terms of integrals (21) with G -functions) for the m -order differentiation and integration (compare with (16)). Namely:

$$D^m = \left(\frac{d}{dx} \right)^m = x^{-m} \left\{ \left(x \frac{d}{dx} \right) \left(x \frac{d}{dx} - 1 \right) \dots \left(x \frac{d}{dx} - m + 1 \right) \right\}; \tag{32}$$

$$I^m f(x) = \int_0^x \frac{(x-t)^{m-1}}{(m-1)!} f(t) dt = x^m \int_0^1 G_{1,1}^{1,0} \left[\sigma \left| \begin{matrix} m \\ 0 \end{matrix} \right. \right] f(x\sigma) d\sigma = x^m I_1^{0,m} f(x),$$

as an E-K “fractional” integral (24), or using a representation of $G_{1,1}^{1,0}$ as a $G_{m,m}^{m,0}$ -function, also as an m -tuple generalized fractional integral as follows:

$$I^m f(x) = \left(\frac{x}{m} \right)^m \int_0^1 G_{m,m}^{m,0} \left[\sigma \left| \begin{matrix} (\frac{k}{m})_1^m \\ (\frac{k}{m} - 1)_1^m \end{matrix} \right. \right] f \left(x\sigma^{1/m} \right) d\sigma = \left(\frac{x}{m} \right)^m I_{m,m}^{(\frac{k}{m}-1), (1)} f(x). \tag{34}$$

The Poisson-Dimovski transformation (18) is also an operator of GFC, with a multi-order of integration (p_1, \dots, p_k) , $p_k := \Delta - \Delta_k = \Delta + \gamma_k - \frac{k}{m} + 1 > 0$, $k = 1, \dots, m$ (see details in [15], [9], [11]), which is essentially used in our manipulations here:

$$\mathcal{P}f(x) = C \left(\frac{x^\beta}{\beta^m} \right)^\Delta I_{\beta, m}^{(\frac{k}{m}-1), (p_k)} f \left(\frac{m}{\beta} x^{\beta/m} \right). \quad (35)$$

4. Solving homogeneous hyper-Bessel equations, the hyper-Bessel functions

Consider now a hyper-Bessel differential operator (3), assuming in addition that $\gamma_m = 0$, and for shortness of denotations $\beta := m$ (the case $\beta \neq m$ is easily done then by a simple substitution $x \mapsto x^{\beta/m}$, as seen in [15]). That is, let

$$\beta = m; \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_m = 0; \alpha := \max_k [-\beta(\gamma_k + 1)] = -m. \quad (36)$$

Then, $\Delta = \max_k \Delta_k = \tilde{\gamma}_m - \gamma_m = 0$, and we get the following *simplified representation of the Poisson-Dimovski transformation* (18) *as a generalized fractional integral* (35), denoted by \mathcal{P}_0 (since $p_m = 0$, its multiplicity is reduced to $(m-1)$), for which $\mathcal{P}_0 : C_{-1} \longrightarrow C_{-1}^{(m)} \subset C_{-m}$:

$$\begin{aligned} \mathcal{P}_0 f(x) &= c I_{m, m-1}^{(\frac{k}{m}-1), (p_k)} f(x) = c \int_0^1 G_{m-1, m-1}^{m-1, 0} \left[\sigma \middle| \begin{matrix} (\gamma_k)_1^{m-1} \\ (\frac{k}{m}-1)_1^{m-1} \end{matrix} \right] f(x\sigma^{1/m}) d\sigma, \\ p_k &= \gamma_k - \frac{k}{m} + 1, \quad k = 1, \dots, m-1. \end{aligned} \quad (37)$$

THEOREM 2. Let $\tilde{y}_\lambda(x)$ be the solution of the initial value problem (IVP) for the simplest m -order ODE with D^m :

$$\left(\frac{d}{dx} \right)^m \tilde{y}(x) = \lambda \tilde{y}(x), \quad \tilde{y}(0) = 1, \quad \tilde{y}'(0) = \dots = \tilde{y}^{(m-1)}(0) = 0. \quad (38)$$

Then, its P - D image $y_\lambda(x) := \mathcal{P}_0 \tilde{y}_\lambda(x)$ under transformation (37), is the solution of the IVP for the hyper-Bessel differential equation

$$B y(x) = \lambda y(x), \quad y(0) = 1, \quad y'(0) = \dots = y^{(m-1)}(0) = 0. \quad (39)$$

P r o o f. According to Corollary 3, \mathcal{P}_0 transmutes D^m into B , being a similarity from D^m to B in $C_{m-1}^{(m)} \subset C_{-1}^{(m)}$: $\mathcal{P}_0 D^m = B \mathcal{P}_0$. We apply the transmutation \mathcal{P}_0 to the equation $\tilde{y}^{(m)}(x) = \lambda \tilde{y}(x)$ and to its initial value conditions. We use that $\tilde{F} \tilde{y}_\lambda = 1$, $\mathcal{P}_0 \{1\} = 1$, $B \{1\} = (x \frac{x}{dx} + m\gamma_1) \dots (x \frac{x}{dx}) \{1\} = 0$, that is $B \mathcal{P}_0 \tilde{F} \tilde{y} = 0$ and $y^{(k)}(0) = (\mathcal{P}_0 \tilde{y})^{(k)}(0) = \delta_{0k}$, where δ_{0k} equals 1 for $k = 0$, and 0 for $k \neq 0$. For details, see Kiryakova [15], Corollary 3.7.1. ■

Now we demonstrate the application of this proposition to find the explicit solution of the IVP (39).

EXAMPLE 1. Let us find the explicit form of the solution of the initial value problem (39) for $\lambda = \pm 1$:

$$By(x) = \pm y(x), \quad y(0) = 1, \quad y'(0) = \dots = y^{(m-1)}(0) = 0. \quad (40)$$

The solution of (38) for $\lambda = -1$, is the *generalized cosine function* (10) of order m (see [12], vol. 3), and we represent it in terms of Meijer's G -function:

$$\begin{aligned} \tilde{y}_{-1}(x) = \cos_m(x) &= \sum_{j=0}^{\infty} \frac{(-1)^j x^{mj}}{(mj)!} = {}_0F_{m-1} \left((k/m)_1^{m-1}; -(x/m)^m \right) \quad (41) \\ &= \sqrt{(2\pi)^{m-1}/m} \, G_{0,m}^{1,0} \left[\left(\frac{x}{m} \right)^m \left| \begin{matrix} - \\ 0, (1 - \frac{k}{m})_1^{m-1} \end{matrix} \right. \right]. \end{aligned}$$

For $y_{-1}(x) = \mathcal{P}_0 \tilde{y}_{-1}(x)$, we apply the representation (37) of \mathcal{P}_0 and use the properties of the G -function ([12], Vol.1) and the formula for integral of product of two G -functions ([12], Vol.1; [15], App.; [20]):

$$\begin{aligned} y_{-1}(x) &= c \int_0^1 G_{m-1,m-1}^{m-1,0} \left[\sigma \left| \begin{matrix} (\gamma_k) \\ (\frac{k}{m} - 1) \end{matrix} \right. \right] \tilde{y}_{-1}(x\sigma^{1/m}) d\sigma \\ &= \prod_{k=1}^{m-1} \Gamma(\gamma_k + 1) \int_0^1 G_{m-1,m-1}^{m-1,0} \left[\sigma \left| \begin{matrix} (\gamma_k) \\ (\frac{k}{m} - 1) \end{matrix} \right. \right] G_{0,m}^{1,0} \left[\left(\frac{x}{m} \right)^m \sigma \left| \begin{matrix} - \\ 0, (1 - \frac{k}{m}) \end{matrix} \right. \right] d\sigma \\ &= \prod_{k=1}^{m-1} \Gamma(\gamma_k + 1) G_{0,m}^{1,0} \left[\left(\frac{x}{m} \right)^m \left| \begin{matrix} - \\ 0, (1 - (\gamma_k + 1)) \end{matrix} \right. \right] = {}_0F_{m-1} \left((\gamma_k + 1); - \left(\frac{x}{m} \right)^m \right). \end{aligned}$$

Thus we obtain the solution $y_{-1}(x)$ as the "normalized" hyper-Bessel function (see def. in [19], [15]):

$$y_{-1}(x) = j_{\gamma_1, \dots, \gamma_{m-1}}^{(m-1)}(x) := {}_0F_{m-1}((\gamma_k + 1); -(x/m)^m). \quad (42)$$

We proceed analogously for $\lambda = 1$, when the *generalized hyperbolic function* of order m ([12], Vol. 3) is the solution of the simpler problem (38):

$$\tilde{y}_1(x) = h_m(x) := {}_0F_{m-1}((k/m)_1^{m-1}; (x/m)^m)$$

and the solution of the hyper-Bessel differential equation $By = y$ in (39) is found explicitly as the "normalized" (by the initial conditions imposed) *modified hyper-Bessel function*:

$$y_1(x) = i_{\gamma_1, \dots, \gamma_{m-1}}^{(m-1)}(x) := {}_0F_{m-1}((\gamma_k + 1); (x/m)^m). \quad (43)$$

DEFINITION 9. The so-called *hyper-Bessel functions*

$$J_{\gamma_1, \dots, \gamma_{m-1}}^{(m-1)}(x) := \frac{(x/m)^{\sum \gamma_k}}{\prod \Gamma(\gamma_k + 1)} {}_0F_{m-1} \left((\gamma_k + 1); -\left(\frac{x}{m}\right)^m \right), \quad (44)$$

$$I_{\gamma_1, \dots, \gamma_{m-1}}^{(m-1)}(x) := \frac{(x/m)^{\sum \gamma_k}}{\prod \Gamma(\gamma_k + 1)} {}_0F_{m-1} \left((\gamma_k + 1); \left(\frac{x}{m}\right)^m \right), \quad (45)$$

were introduced by Delerue [2] and studied later also by Ključantčev [19], and Dimovski and Kiryakova [10]; [15], Chs. 3 and 4; [16], etc. as *multi-index analogues of the Bessel functions* $J_\nu(x)$, $I_\nu(x)$.

NOTE. The P-D transmutation is applied also to solve *non-homogeneous* hyper-Bessel differential equations (8) with $f \neq 0$, see e.g. in [15], §3.8.

5. Examples of solving hyper-Bessel differential equations by means of the differential Poisson-Dimovski transformation

The Poisson-Dimovski transformation (18), (35) can be considered not only as an integral transform, but for suitable choice of its parameters - *also as a differential transform*. In this case, the definition of the generalized fractional derivatives $D_{\beta, m}^{(\gamma_k), (\delta_k)}$ is used as a suitable interpretation of the generalized fractional calculus operator in (35), when the multi-order of "integration" includes also negative components δ_k , see Definition 8.

Especially, if all the components $\delta_k = \eta_k$, $k = 1, \dots, m$ are positive integers, then $I_{\beta, m}^{(\gamma_k + \delta_k), (\eta_k - \delta_k)} = Id$ (the identity operator) in definition (28), and $D_{\beta, m}^{(\gamma_k), (\delta_k)}$ becomes a purely differential operator

$$D_\eta = \prod_{k=1}^m \prod_{j=1}^{\eta_k} \left(\frac{1}{\beta} x \frac{d}{dx} + \tilde{\gamma}_k - \eta_k + j \right). \quad (46)$$

Let us consider the Poisson-Dimovski transformation (18), (35) but instead of $\Delta = \max_k \Delta_k$, we take $\Delta = \min_k \Delta_k$. Then, $p_k = -\eta_k = \Delta - \Delta_k \leq 0$, $k = 1, \dots, m$, and the symbol

$$\mathcal{P} = \left(\frac{x^\beta}{\beta^m} \right)^\Delta I_{\beta, m}^{(\frac{k}{m}-1), (-\eta_k)} = \left(\frac{x^\beta}{\beta^m} \right)^\Delta D_{\beta, m}^{(\frac{k}{m}-1-\eta_k), (\eta_k)} \quad (47)$$

has to be interpreted as an operator for generalized fractional differentiation.

Let B be a "spherical" hyper-Bessel differential operator (3) for which

$$\beta = m; \quad \gamma_k = \frac{k}{m} - 1 - \eta_k, \quad k = 1, \dots, m-1; \quad \gamma_m = -\eta_m = 0, \quad (48)$$

with "semi-integer" γ_k , i.e. integer $\eta_k \geq 0$, $k = 1, \dots, m-1$.

Then, all the differences $\Delta_k = \tilde{\gamma}_k - \gamma_k$ are nonnegative integers, and at least of them is zero ($\Delta_m = 0$). Let us choose $\Delta = \min_k \Delta_k = 0$, that

is, $p_k = -\delta_k = -\eta_k \leq 0$, $k = 1, \dots, m-1$; $p_m = 0$. In this case, the Poisson-Dimovski transformation is represented by means of the *differential operator*

$$\begin{aligned} \mathcal{P} &= I_{m,m-1}^{(\tilde{\gamma}_k),(-\eta_k)} = D_{m,m-1}^{(\gamma_k),(\frac{k}{m}-1-\gamma_k)} := D_\eta \\ &= \prod_{k=1}^{m-1} \prod_{j=1}^{\eta_k} \left(\frac{1}{m} x \frac{d}{dx} + \tilde{\gamma}_k - \eta_k + j \right) = \prod_{k=1}^{m-1} \prod_{j=1}^{\eta_k} \left(\frac{1}{m} x \frac{d}{dx} + \gamma_k + j \right). \end{aligned} \quad (49)$$

DEFINITION 10. The transformation $D_\eta : C_\alpha^{(\eta_1+\dots+\eta_{m-1})} \longrightarrow C_\alpha$ defined by the differential operator

$$D_\eta f(x) = \prod_{k=1}^{m-1} \prod_{j=1}^{\eta_k} \left(x^m \frac{d}{d(x^m)} + \gamma_k + j \right) \quad (50)$$

we call as *Poisson-Dimovski differential transformation*, corresponding to the "spherical" hyper-Bessel differential operator B with parameters (48).

Note that in particular, $D_\eta : C_a^{(\infty)} \longrightarrow C_a^{(\infty)}$. Let $\alpha = \max_k [-\beta(\gamma_k + 1)] = m \max_k (\eta_k - \frac{k}{m})$, and $\tilde{\mathcal{X}} := C_{-1}^{(\infty)} = C_\alpha^{(\infty)}$. If we choose $a = \max(\alpha, -1)$, then $\hat{\mathcal{X}} = C_a^{(\infty)} = \tilde{\mathcal{X}} \cap \mathcal{X}$ and $D_\eta : \hat{\mathcal{X}} \longrightarrow \hat{\mathcal{X}}$.

THEOREM 3. The differential Poisson-Dimovski (P-D) transform (50) is a similarity from the m -fold integration I^m to the hyper-Bessel integral operator L in the functional space $C_{\max(\alpha, -1)}^{(\eta_1+\dots+\eta_m)}$:

$$D_\eta I^m = L D_\eta. \quad (51)$$

P r o o f. We use the representations

$$D_\eta = I_{m,m-1}^{(\tilde{\gamma}_k),(-\eta_k)}, \quad I^m = \left(\frac{x}{m} \right)^m I_{m,m-1}^{(\tilde{\gamma}_k),(1)}, \quad L = \left(\frac{x}{m} \right)^m I_{m,m-1}^{(\gamma-k),(1)}.$$

Then according to the operational rules for the generalized fractional calculus operators, see [18], p. 214 - Th.2, (34) and (37):

$$\begin{aligned} I_{\beta,m}^{(\gamma_k),(\delta_k)} x^{\beta p} f(x) &= x^{\beta p} I_{\beta,m}^{(\gamma_k+p),(\delta_k)} f(x), \\ I_{\beta,m}^{(\gamma_k+\delta_k),(\sigma_k)} I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) &= I_{\beta,m}^{(\gamma_k),(\sigma_k+\delta_k)} f(x) \text{ semigroup property,} \end{aligned}$$

we obtain

$$I_{m,m-1}^{(\tilde{\gamma}_k),(-\eta_k)} \left(\frac{x}{m} \right)^m I_{m,m-1}^{(\tilde{\gamma}_k),(1)} f(x) = \left(\frac{x}{m} \right)^m I_{m,m-1}^{(\tilde{\gamma}_k+1),(1)} I_{m,m-1}^{(\tilde{\gamma}_k),(1)} f(x)$$

$$= \left(\frac{x}{m}\right)^m I_{m,m-1}^{(\gamma_k), (1)} I_{m,m-1}^{(\tilde{\gamma}_k), (-\eta_k)} f(x),$$

that is, $D_\eta I^m f(x) = L D_\eta f(x)$. For more details, see in [15], §3.5. \blacksquare

COROLLARY 4. *If $\tilde{u} \in C_{m+\max(\alpha, -1)}^{(\infty)}$ is a solution of the m -th order differential equation $\tilde{u}^{(m)}(x) = \lambda \tilde{u}(x)$, then the P-D image $u = D_\eta \tilde{u} \in C_{m+\max(\alpha, -1)}^{(\infty)}$ is a solution of the hyper-Bessel differential equation $Bu(x) = \lambda u(x)$ with parameters as in (48).*

Let us consider now some illustrations of the use of the P-D differential transformation. We use some *examples of ODEs of hyper-Bessel type taken from the popular Kamke's book* [14].

EXAMPLE 2. Consider the ODE

$$x y^{(m)}(x) - n m y^{(m-1)}(x) + a x y(x) = 0, \quad 0 < x < \infty, \quad (52)$$

where m and n are natural numbers, $m \geq 2$, $a = \text{const}$. It can be written also in the form $B y(x) = -y(x)$ with the m -th order hyper-Bessel operator $B = x^{-m} Q_m(x \frac{d}{dx})$, where: $\beta = m$, $\gamma_1 = \frac{1}{m} - (n+1)$; $\gamma_k = \frac{k}{m} - 1 := \tilde{\gamma}_k$, $k = 2, \dots, m$. Therefore, $\Delta_1 = \eta - 1 = n > 0$ - an integer, and $\Delta_k = \eta_k = 0$, $k = 2, \dots, m$. Then the corresponding P-D transformation (50) has the form of the differential operator

$$D_\eta f(x) = \prod_{j=1}^n \left(x^m \frac{d}{d(x^m)} + \gamma_1 + j \right) f(x) = \left[x^{(n+1)m-1} \left(x^{1-m} \frac{d}{dx} \right)^n x^{1-m} \right] f(x),$$

and, according to Corollary 4, if $\tilde{y}(x)$ denotes a solution of the simpler equation $\tilde{y}^{(m)}(x) + a y(x) = 0$, then the P-D image

$$y(x) = D_\eta \tilde{y}(x) = \left[x^{(n+1)m-1} \left(x^{1-m} \frac{d}{dx} \right)^n x^{1-m} \right] \tilde{y}(x) \quad (53)$$

is a solution of eq. (52). The same result is given in Kamke [14], p. 483, for *Problem 5.6*.

EXAMPLE 3. Consider the 3-rd order ODE equation

$$x^2 y'''(x) - 3(p+q)xy''(x) + 3p(3q+1)y'(x) - x^2 y(x) = 0, \quad 0 < x < \infty, \quad (54)$$

with integers $p > 0, q > 0$. It is a hyper-Bessel equation of the form $B y = y$:

$$B = x^{-3} Q_3(x \frac{d}{dx}) \quad \text{with} \quad m = \beta = 3; \quad \gamma_1 = q - \frac{2}{3}, \gamma_2 = -p - \frac{1}{3}, \gamma_3 = 0.$$

Then the considered differences are: $\Delta_1 = \eta_1 = q, \Delta_2 = \eta_2 = p, \Delta_3 = 0$, and the P-D transformation (50) has the form

$$\begin{aligned}
D_\eta &= 3^{(p+q)} \prod_{k=1}^3 \prod_{j'=1}^{\eta_k} \left(x^3 \frac{d}{d(x^3)} + \gamma_k + j' \right) \\
&= 3^{(p+q)} \prod_{j_1=0}^{p-1} \left(\frac{1}{3} x \frac{d}{dx} - j_1 - \frac{1}{3} \right) \prod_{j_2=0}^{q-1} \left(\frac{1}{3} x \frac{d}{dx} - j_2 - \frac{2}{3} \right) \\
&= \prod_{j_1=0}^{p-1} \left(x \frac{d}{dx} - 3j_1 - 1 \right) \prod_{j_2=0}^{q-1} \left(x \frac{d}{dx} - 3j_2 - 2 \right).
\end{aligned}$$

If $\tilde{y}(x)$ denotes a solution of the simpler 3-rd order differential equation $\tilde{y}''' = \tilde{y}$, that is $\tilde{y}(x) = \sum_{k=1}^3 c_k \exp(\omega_k x)$, $\omega_k^3 = 1$, then as it also shown in Kamke [14], p. 466, *Problem 3.49*, we can find the solution of (54) in the form

$$y(x) = D_\eta \tilde{y}(x) = \prod_{j_1=0}^{p-1} \left(x \frac{d}{dx} - 3j_1 - 1 \right) \prod_{j_2=0}^{q-1} \left(x \frac{d}{dx} - 3j_2 - 2 \right) \tilde{y}(x).$$

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